ON THE OSTROWSKI TYPE INTEGRAL INEQUALITY FOR DOUBLE INTEGRALS

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Abstract. In this note, we establish new an inequality of Ostrowski-type for double integrals involving functions of two independent variables by using fairly elementary analysis.

1. Introduction

In 1938, the classical integral inequality established by Ostrowski [3] as follows:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f^{'}:(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty=\sup\limits_{t\in(a,b)}|f'(t)|<\infty$. Then we have the inequality:*

$$\left|f(x)-\frac{1}{b-a}\int\limits_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

In a recent paper [1], Barnett and Dragomir proved the following Ostrowski type inequality for double integrals

Theorem 2. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be continuous on $[a,b]\times[c,d]$, $f''_{x,y}=\frac{\partial^2 f}{\partial x\partial y}$ exists on $(a,b)\times(c,d)$ and is bounded, i.e., $\left\|f''_{x,y}\right\|_\infty=\sup\limits_{(x,y)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right|<\infty$. Then, we have the inequality:*

$$\begin{aligned}&\left|\int\limits_a^b\int\limits_c^d f(s,t)dtds-(d-c)(b-a)f(x,y)\right.\\ &\left.-\left[(b-a)\int\limits_c^d f(x,t)dt+(d-c)\int\limits_a^b f(s,y)ds\right]\right|\\ \leq&\left[\frac{1}{4}(b-a)^2+(x-\frac{a+b}{2})^2\right]\left[\frac{1}{4}(d-c)^2+(y-\frac{d+c}{2})^2\right]\left\|f''_{x,y}\right\|_\infty\end{aligned}\tag{1.2}$$

for all $(x,y)\in[a,b]\times[c,d]$.

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In [1], the inequality (1.2) is established by the use of integral identity involving Peano kernels. In [5], Pachpatte obtained an inequality in the view (1.2) by using elementary analysis. The interested reader is also refered to ([1], [2], [4]-[8]) for Ostrowski type inequalities in several independent variables.

The main aim of this note is to establish a new Ostrowski type inequality for double integrals involving functions of two independent variables and their partial derivatives.

2. Main Result

Theorem 3. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be an absolutely continuous fuction such that the partial derivative of order 2 exist and is bounded, i.e.,*

$$\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty=\sup_{(x,y)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right|<\infty$$

for all $(t,s)\in[a,b]\times[c,d]$. Then, we have

$$\begin{aligned}
&\Big|(\beta_1-\alpha_1)(\beta_2-\alpha_2)f(\tfrac{a+b}{2},\tfrac{c+d}{2})+H(\alpha_1,\alpha_2,\beta_1,\beta_2)+G(\alpha_1,\alpha_2,\beta_1,\beta_2)\\
&-(\beta_2-\alpha_2)\int_a^b f(t,\tfrac{c+d}{2})dt-(\beta_1-\alpha_1)\int_c^d f(\tfrac{a+b}{2},s)ds\\
&-\int_a^b[(\alpha_2-c)f(t,c)+(d-\beta_2)f(t,d)]dt-\int_c^d[(\alpha_1-a)f(a,s)+(b-\beta_1)f(b,s)]ds\\
&+\int_a^b\int_c^d f(t,s)dsdt\Big|\leq\left[\frac{(\alpha_1-a)^2+(b-\beta_1)^2}{2}+\frac{(a+b-2\alpha_1)^2+(a+b-2\beta_1)^2}{8}\right]\\
&\times\left[\frac{(\alpha_2-c)^2+(d-\beta_2)^2}{2}+\frac{(c+d-2\alpha_2)^2+(c+d-2\beta_2)^2}{8}\right]\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty
\end{aligned}\tag{2.1}$$

for all $(\alpha_1,\alpha_2),(\beta_1,\beta_2)\in[a,b]\times[c,d]$ with $\alpha_1<\beta_1,\ \alpha_2<\beta_2$ where

$$\begin{aligned}
H(\alpha_1,\alpha_2,\beta_1,\beta_2)&=(\alpha_1-a)[(\alpha_2-c)f(a,c)+(d-\beta_2)f(a,d)]\\
&+(b-\beta_1)[(\alpha_2-c)f(b,c)+(d-\beta_2)f(b,d)]
\end{aligned}\tag{2.2}$$

and

$$\begin{aligned}
G(\alpha_1,\alpha_2,\beta_1,\beta_2)&=(\beta_1-\alpha_1)\left[(\alpha_2-c)f(\tfrac{a+b}{2},c)+(d-\beta_2)f(\tfrac{a+b}{2},d)\right]\\
&+(\beta_2-\alpha_2)\left[(\alpha_1-a)f(a,\tfrac{c+d}{2})+(b-\beta_1)f(b,\tfrac{c+d}{2})\right].
\end{aligned}\tag{2.3}$$

Proof. We define the following functions:

$$p(a,b,\alpha_1,\beta_1,t)=\begin{cases} t-\alpha_1, & t\in[a,\frac{a+b}{2}]\\ t-\beta_1, & t\in(\frac{a+b}{2},b]\end{cases}$$

and

$$q(c,d,\alpha_2,\beta_2,s) = \begin{cases} s-\alpha_2, & s\in[c,\frac{c+d}{2}] \\ s-\beta_2, & s\in(\frac{c+d}{2},d] \end{cases}$$

for all $(\alpha_1,\alpha_2),(\beta_1,\beta_2)\in[a,b]\times[c,d]$ with $\alpha_1<\beta_1,\ \alpha_2<\beta_2$. Thus, by definitions of $p(a,b,\alpha_1,\beta_1,t)$ and $q(c,d,\alpha_2,\beta_2,s)$, we have
(2.4)

$$\int_a^b\int_c^d p(a,b,\alpha_1,\beta_1,t)q(c,d,\alpha_2,\beta_2,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt = \int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}(t-\alpha_1)(s-\alpha_2)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt$$

$$+\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d(t-\alpha_1)(s-\beta_2)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt+\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}}(t-\beta_1)(s-\alpha_2)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt$$

$$+\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d(t-\beta_1)(s-\beta_2)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt.$$

Integrating by parts, we can state:
(2.5)

$$\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}(t-\alpha_1)(s-\alpha_2)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt = \frac{(a+b-2\alpha_1)(c+d-2\alpha_2)}{4}f(\tfrac{a+b}{2},\tfrac{c+d}{2})+\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}f(t,s)dsdt$$

$$-\frac{(a-\alpha_1)(c+d-2\alpha_2)}{2}f(a,\tfrac{c+d}{2})-\frac{(a+b-2\alpha_1)(c-\alpha_2)}{2}f(\tfrac{a+b}{2},c)+(a-\alpha_1)(c-\alpha_2)f(a,c)$$

$$-\int_a^{\frac{a+b}{2}}[\frac{(c+d-2\alpha_2)}{2}f(t,\tfrac{c+d}{2})-(c-\alpha_2)f(t,c)]dt-\int_c^{\frac{c+d}{2}}[\frac{(a+b-2\alpha_1)}{2}f(\tfrac{a+b}{2},s)-(a-\alpha_1)f(a,s)]ds.$$

(2.6)

$$\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d(t-\alpha_1)(s-\beta_2)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt = -\frac{(a+b-2\alpha_1)(c+d-2\beta_2)}{4}f(\tfrac{a+b}{2},\tfrac{c+d}{2})+\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d f(t,s)dsdt$$

$$+\frac{(a-\alpha_1)(c+d-2\beta_2)}{2}f(a,\tfrac{c+d}{2})+\frac{(a+b-2\alpha_1)(d-\beta_2)}{2}f(\tfrac{a+b}{2},d)+(a-\alpha_1)(d-\beta_2)f(a,d)$$

$$+\int_a^{\frac{a+b}{2}}[\frac{(c+d-2\beta_2)}{2}f(t,\tfrac{c+d}{2})-(d-\beta_2)f(t,d)]dt-\int_{\frac{c+d}{2}}^d[\frac{(a+b-2\alpha_1)}{2}f(\tfrac{a+b}{2},s)-(a-\alpha_1)f(a,s)]ds.$$

(2.7)
$$\int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}}(t-\beta_1)(s-\alpha_2)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt=-\frac{(a+b-2\beta_1)(c+d-2\alpha_2)}{4}f(\tfrac{a+b}{2},\tfrac{c+d}{2})+\int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}}f(t,s)dsdt$$

$$+\frac{(b-\beta_1)(c+d-2\alpha_2)}{2}f(b,\tfrac{c+d}{2})+\frac{(a+b-2\beta_1)(c-\alpha_2)}{2}f(\tfrac{a+b}{2},c)-(b-\beta_1)(c-\alpha_2)f(b,c)$$

$$-\int_{\frac{a+b}{2}}^{b}[\frac{(c+d-2\alpha_2)}{2}f(t,\tfrac{c+d}{2})-(c-\alpha_2)f(t,c)]dt+\int_{c}^{\frac{c+d}{2}}[\frac{(a+b-2\beta_1)}{2}f(\tfrac{a+b}{2},s)-(b-\beta_1)f(b,s)]ds.$$

(2.8)
$$\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}(t-\beta_1)(s-\beta_2)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt=\frac{(a+b-2\beta_1)(c+d-2\beta_2)}{4}f(\tfrac{a+b}{2},\tfrac{c+d}{2})+\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}f(t,s)dsdt$$

$$-\frac{(b-\beta_1)(c+d-2\beta_2)}{2}f(b,\tfrac{c+d}{2})-\frac{(a+b-2\beta_1)(d-\beta_2)}{2}f(\tfrac{a+b}{2},d)+(b-\beta_1)(d-\beta_2)f(b,d)$$

$$+\int_{\frac{a+b}{2}}^{b}[\frac{(c+d-2\beta_2)}{2}f(t,\tfrac{c+d}{2})-(d-\beta_2)f(t,d)]dt+\int_{\frac{c+d}{2}}^{d}[\frac{(a+b-2\beta_1)}{2}f(\tfrac{a+b}{2},s)-(b-\beta_1)f(b,s)]ds.$$

Adding (2.5)-(2.8) and rewriting, we easily deduce:
(2.9)
$$\int_{a}^{b}\int_{c}^{d}p(a,b,\alpha_1,\beta_1,t)q(c,d,\alpha_2,\beta_2,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt=(\beta_1-\alpha_1)(\beta_2-\alpha_2)f(\tfrac{a+b}{2},\tfrac{c+d}{2})+H(\alpha_1,\alpha_2,\beta_1,\beta_2)$$

$$+G(\alpha_1,\alpha_2,\beta_1,\beta_2)-(\beta_2-\alpha_2)\int_{a}^{b}f(t,\tfrac{c+d}{2})dt-(\beta_1-\alpha_1)\int_{c}^{d}f(\tfrac{a+b}{2},s)ds$$

$$-\int_{a}^{b}[(\alpha_2-c)f(t,c)+(d-\beta_2)f(t,d)]dt-\int_{c}^{d}[(\alpha_1-a)f(a,s)+(b-\beta_1)f(b,s)]ds$$

$$+\int_{a}^{b}\int_{c}^{d}f(t,s)dsdt$$

where $H(\alpha_1, \alpha_2, \beta_1, \beta_2)$ and $G(\alpha_1, \alpha_2, \beta_1, \beta_2)$ defined by (2.2) and (2.3), respectively. Now, using the identitiy (2.9), it follows that

(2.10)

$$\Bigg|(\beta_1 - \alpha_1)(\beta_2 - \alpha_2) f(\tfrac{a+b}{2}, \tfrac{c+d}{2}) + H(\alpha_1, \alpha_2, \beta_1, \beta_2) + \int_a^b \int_c^d f(t, s) ds dt$$

$$+G(\alpha_1, \alpha_2, \beta_1, \beta_2) - (\beta_2 - \alpha_2) \int_a^b f(t, \tfrac{a+b}{2}) dt - (\beta_1 - \alpha_1) \int_c^d f(x, \tfrac{c+d}{2}) ds$$

$$- \int_a^b [(\alpha_2 - c) f(t, c) + (d - \beta_2) f(t, d)] dt - \int_c^d [(\alpha_1 - a) f(a, s) + (b - \beta_1) f(b, s)] ds \Bigg|$$

$$\leq \int_a^b \int_c^d |p(a, b, \alpha_1, \beta_1, t)| \, |q(c, d, \alpha_2, \beta_2, s)| \left| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right| ds dt$$

$$\leq \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_\infty \int_a^b \int_c^d |p(a, b, \alpha_1, \beta_1, t)| \, |q(c, d, \alpha_2, \beta_2, s)| \, ds dt.$$

On the other hand, we get

$$\begin{aligned}
\int_a^b |p(a, b, \alpha_1, \beta_1, t)| \, dt &= \int_a^{\frac{a+b}{2}} |t - \alpha_1| \, dt + \int_{\frac{a+b}{2}}^b |t - \beta_1| \, dt \\
&= \int_a^{\alpha_1} (\alpha_1 - t) \, dt + \int_{\alpha_1}^{\frac{a+b}{2}} (t - \alpha_1) \, dt + \int_{\frac{a+b}{2}}^{\beta_1} (\beta_1 - t) \, dt + \int_{\beta_1}^b (t - \beta_1) \, dt \\
&= \frac{(\alpha_1 - a)^2 + (b - \beta_1)^2}{2} + \frac{(a + b - 2\alpha_1)^2 + (a + b - 2\beta_1)^2}{8} \qquad (2.11)
\end{aligned}$$

and similarly,

$$\begin{aligned}
\int_c^d |q(a, b, \alpha_1, \beta_1, t)| \, dt &= \int_c^{\frac{c+d}{2}} |s - \alpha_2| \, ds + \int_{\frac{c+d}{2}}^d |s - \beta_2| \, ds \\
&= \frac{(\alpha_2 - c)^2 + (d - \beta_2)^2}{2} + \frac{(c + d - 2\alpha_2)^2 + (c + d - 2\beta_2)^2}{8}. \qquad (2.12)
\end{aligned}$$

Using (2.11) and (2.12) in (2.10), we see that (2.1) holds. □

Corollary 1. *Under the assumptions of Theorem 3, we have*

$$\left|(b-a)(d-c)f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-(d-c)\int_a^b f\left(t,\frac{c+d}{2}\right)dt-(b-a)\int_c^d f\left(\frac{a+b}{2},s\right)ds\right.$$
$$\left.+\int_a^b\int_c^d f(t,s)dsdt\right|\leq\frac{1}{16}\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty (b-a)^2(d-c)^2. \tag{2.13}$$

Proof. We choose $\alpha_1=a,\ \beta_1=b,\ \alpha_2=c$ and $\beta_2=d$ in (2.1), then we see that (2.13) holds. □

Corollary 2. *Under the assumptions of Theorem 3, we have*

$$\left|\frac{(b-a)(d-c)}{4}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right]-\frac{(d-c)}{2}\int_a^b\left[f(t,c)+f(t,d)\right]dt\right.$$
$$\left.-\frac{(b-a)}{2}\int_c^d\left[f(a,s)+f(b,s)\right]ds+\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\leq\frac{1}{16}\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty (b-a)^2(d-c)^2. \tag{2.14}$$

Proof. We choose $\alpha_1=\beta_1=\frac{a+b}{2},\ \alpha_2=\beta_2=\frac{c+d}{2}$ in (2.1), then we see that (2.14) holds. □

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